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THE ESSENTIAL RANGE OF A NONABELIAN COCYCLE IS NOT A COHOMOLOGY INVARIANT

BY

L. ARNOLD

Institute for Dynamical Systems, University of Bremen
P. O. Box 330 440, 28334 Bremen, Germany
email: arnold@math.uni-bremen.de

AND

NGUYEN DINH CONG

Hanoi Institute of Mathematics, P. O. Box 631 Bo Ho, 10000 Hanoi, Vietnam
email: cong@thevinh.ac.vn

AND

V. I. OSELEDETS

Department of Mechanics and Mathematics, Moscow State University
Moscow 119 899, Russia
email: oseled@mech.math.msu.su

ABSTRACT

We show by way of examples that the essential range of a nonabelian cocycle is in general not invariant under cocycle cohomology, and differs in general from the essential range of an induced cocycle.

1. Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a non-atomic probability space, and let θ be an ergodic automorphism of $(\Omega, \mathcal{F}, \mathbb{P})$ preserving the probability measure $\mathbb{P}$.

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A measurable map $A(\cdot)\colon \Omega \to Gl(d)$ from the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to the general linear group $Gl(d)$ equipped with its Borel σ-algebra generates a *linear cocycle* over the dynamical system (Ω, θ) via

$$\Phi_A(n,\omega) := \begin{cases} A(\theta^{n-1}\omega)\dots A(\omega), & n>0,\\ I, & n=0,\\ A^{-1}(\theta^n\omega)\dots A^{-1}(\theta^{-1}\omega), & n<0. \end{cases}$$

Conversely, if we are given a linear cocycle over θ, then its time-one map is a measurable linear map. Therefore, the correspondence between A and Φ_A is one-to-one so that we can identify Φ_A with A and speak of the cocycle A.

The above construction applies to any measurable group G in place of $Gl(d)$, and we shall then speak of a G-cocycle and a G-map.

Since we deal with discrete time cocycles we can always neglect sets of null measure, and often omit the $\mathbb{P}$-almost sure statement in equations between measurable functions.

Two G-cocycles A and B are called G-**cohomologous** if there exists a G-map C such that for almost all $\omega \in \Omega$

$$B(\omega) = C(\theta\omega)^{-1} \circ A(\omega) \circ C(\omega).$$

In this case C is called a G-**cohomology** and we write $A \sim B$. A G-cocycle which is G-cohomologous to the trivial G-cocycle, i.e. the cocycle identically equal to the identity of G, is called a G-**coboundary**.

The following notion of essential range was introduced by Araki and Woods [1] and Krieger [4] for studying Radon–Nikodym cocycles, continued by Schmidt [5] for abelian cocycles of $\mathbb{Z}$-actions, and then developed by Feldman and Moore [3] and Schmidt [6] for general cocycles of countable group actions and equivalence relations.

Definition: Let G be a locally compact topological group. The **essential range** of a G-cocycle A is the set $\overline{\mathcal{E}}(A) \subset \overline{G}$, where $\overline{G}$ is the one-point compactification of G ($\overline{G} = G$ if G is compact), consisting of those elements $M \in \overline{G}$ such that for any neighborhood $N(M)$ of M in $\overline{G}$ and any set $E \in \mathcal{F}$ with $\mathbb{P}(E) > 0$ there exist $\omega \in E$ and $n \in \mathbb{Z}$ such that $\theta^n\omega \in E$ and $\Phi_A(n,\omega) \in N(M)$.

The following facts are well-known:

(i) The set $\mathcal{E}(A) := \overline{\mathcal{E}}(A) \cap G$ is a closed subgroup of G.

(ii) If A is G-cohomologous to a G-cocycle B, then $\infty \in \overline{\mathcal{E}}(A)$ if and only if $\infty \in \overline{\mathcal{E}}(B)$.

(iii) A $Gl(d)$-cocycle is cohomologous to an orthogonal cocycle if and only if $\infty \notin \overline{\mathcal{E}}(A)$.

Let X be a complete metric space and G a locally compact topological group. A **continuous action** of G on X is a group homeomorphism from G into the group Homeo(X) of homeomorphisms of X such that the map $G \times X \to X$, $(g,x) \mapsto gx$, is continuous. In this case we call X a G-**space**.

LEMMA 1.1: *Let G be a locally compact topological group, X a G-space, A a G-cocycle and $f\colon \Omega \to X$ an A-invariant function, i.e. a measurable function which satisfies $A(\omega)f(\omega) = f(\theta\omega)$ for almost all $\omega \in \Omega$. Then there exists a set $\Omega_1 \in \mathcal{F}$ of full $\mathbb{P}$-measure such that for any $\omega \in \Omega_1$ and any $M \in \mathcal{E}(A)$*

$$Mf(\omega) = f(\omega).$$

Proof: Since X is a separable metric space, the support of f, i.e. the set

$$\Omega_1 := \left\{\omega \in \Omega \mid \mathbb{P}(\{\omega' \in \Omega \mid \rho(f(\omega'), f(\omega)) < \varepsilon) > 0 \quad \text{for any} \quad \varepsilon > 0\right\},$$

has full $\mathbb{P}$-measure.

Fix an arbitrary $\omega \in \Omega_1$ and an arbitrary $M \in \mathcal{E}(A)$. Let $\varepsilon > 0$ be arbitrary. Then the set $E \subset \Omega$ of those $\omega' \in \Omega$ such that $\rho(f(\omega'), f(\omega)) < \varepsilon$ has positive $\mathbb{P}$-measure. Since $M \in \mathcal{E}(A)$, by the definition of $\mathcal{E}(A)$, for any $\varepsilon_1 > 0$ there exist $\omega_1 \in E$ and $n \in \mathbb{Z}$ such that $\theta^n\omega_1 \in E$ and

$$\|M - \Phi_A(n, \omega_1)\| < \varepsilon_1.$$

Since f is A-invariant,

$$\begin{aligned}\rho(Mf(\omega), f(\omega)) \le \rho(Mf(\omega), Mf(\omega_1)) &+ \rho(Mf(\omega_1), \Phi_A(n,\omega_1)f(\omega_1)) \\ &+ \rho(f(\theta^n\omega_1), f(\omega)).\end{aligned}$$

Since ε and ε_1 are arbitrary and the action of G on X is continuous, we obtain $\rho(Mf(\omega), f(\omega)) = 0$. ∎

2. The essential range is not a cohomology invariant

We settle the problem posed by Feldman and Moore [3] whether the conjugacy class of $\overline{\mathcal{E}}(A)$ is a cohomology invariant in the nonabelian case to the negative.

First note that assuming $A \sim B$ the condition that $\overline{\mathcal{E}}(A)$ is conjugate to $\overline{\mathcal{E}}(B)$ is equivalent to the condition that $\mathcal{E}(A)$ is conjugate to $\mathcal{E}(B)$.

THEOREM 2.1: *There are $Gl(2)$-cocycles A and B such that $A \sim B$ but $\overline{\mathcal{E}}(A)$ is not conjugate to $\overline{\mathcal{E}}(B)$.*

Proof: Choose a $Gl(2)$-cocycle $A : \Omega \to Gl(2)$ of the form

$$A(\omega) := \begin{pmatrix} 1 & a(\omega) \\ 0 & 1 \end{pmatrix}$$

in such a way that $\mathcal{E}(A)$ is nontrivial. This can be achieved by taking an appropriate recurrent additive function $a(\omega)$ (note that this case is equivalent to the one-dimensional abelian case investigated by Dekking [2] and Schmidt [5]).

Choose a measurable map $C\colon \Omega \to SO(2)$ such that the function $\omega \mapsto C(\omega)e_1 \in S^1$, where e_1 is the first standard basis vector, has support equal to S^1. This is possible because Ω is non-atomic.

Put $B(\omega) := C(\theta\omega)A(\omega)C(\omega)^{-1}$ for all $\omega \in \Omega$, hence $B \sim A$. Since the deterministic point $e_1 \in S^1$ is A-invariant, the function $\omega \mapsto C(\omega)e_1 \in S^1$ is B-invariant. By Lemma 1.1, for any $b \in \mathcal{E}(B)$, the equality $bC(\omega)e_1 = C(\omega)e_1$ holds almost surely. Since $C(\omega)e_1$ has support S^1 this is the case if and only if $b = \alpha I$ for some $\alpha \in \mathbb{R} \setminus \{0\}$. On the other hand, $\det b = 1$ because $\det B(\omega) = 1$ for all $\omega \in \Omega$, hence $b = I$. Therefore $\mathcal{E}(B) = \{I\}$ which is not conjugate to $\mathcal{E}(A)$. ∎

It is easily seen that $\overline{\mathcal{E}}(B) = \{I, \infty\}$ for the cocycle B constructed in the proof of Theorem 2.1.

THEOREM 2.2: *There are $SO(3)$-cocycles A and B such that $A \sim B$ but $\overline{\mathcal{E}}(A)$ is not conjugate to $\overline{\mathcal{E}}(B)$.*

Proof: Choose an $SO(3)$-cocycle A of the form

$$A(\omega) = \begin{pmatrix} 1 & 0 \\ 0 & A'(\omega) \end{pmatrix}$$

with A' being an $SO(2)$-cocycle with nontrivial $\mathcal{E}(A')$. Choose a measurable map $C\colon \Omega \to SO(3)$ such that the function $\omega \mapsto C(\omega)e_1$, where e_1 is the first standard basis vector, has support equal to S^2. Put $B(\omega) := C(\theta\omega)A(\omega)C(\omega)^{-1}$ for all $\omega \in \Omega$. Then $B \sim A$. The same argument as in the proof of Theorem 2.1 implies that $\mathcal{E}(B) = \{I\}$. ∎

We also note that Proposition 2.1(1) of Schmidt [6] asserting that a cocycle is a coboundary if and only if its essential range is trivial is false in the nonabelian case. Indeed, the $SO(3)$-cocycle B constructed in the proof of Theorem 2.2 has trivial essential range but it is not a coboundary. For, the orthogonal cocycle A constructed in the proof of Theorem 2.2 is cohomologous to B and has nontrivial essential range. If B is a coboundary then so is A, hence A is cohomologous to the trivial cocycle which obviously is minimal, entailing that its essential range is trivial.

3. The essential range of induced cocycles

Let $E \in \mathcal{F}$ and $\mathbb{P}(E) > 0$. Then the **return time** of E is defined by

$$k_E(\omega) = \begin{cases} \min\{n \geq 1 \mid \theta^n\omega \in E\}, & \text{if } \theta^n\omega \in E \text{ for some } n \in \mathbb{N}, \\ \infty, & \text{if } \theta^n\omega \notin E \text{ for all } n \in \mathbb{N}. \end{cases}$$

By the Poincaré recurrence theorem $k_E(\cdot)$ is finite almost surely. The induced automorphism θ_E of θ on E is defined as

$$\theta_E(\omega) := \theta^{k_E(\omega)}\omega \quad \text{for } \omega \in G,$$

and the probability space $(E, \mathcal{F}_E, \mathbb{P}_E)$ is given by

$$\mathcal{F}_E := \{A \in \mathcal{F} \mid A \subseteq E\}, \quad \mathbb{P}_E(F) := \frac{\mathbb{P}(F)}{\mathbb{P}(E)} \quad \text{for all } F \in \mathcal{F}_E.$$

Let A be a G-cocycle; then the **induced cocycle** over E of A is by definition the G-cocycle $A_E(\omega) := \Phi_A(k_E(\omega), \omega)$, $\omega \in E$, over the induced dynamical system $(E, \mathcal{F}_E, \mathbb{P}_E, \theta_E)$.

It is well-known that for any $E \in \mathcal{F}$ and $\mathbb{P}(E) > 0$, A is cohomologous to B if and only if A_E is cohomologous to B_E.

Further, if the group G is abelian, then for any $E \in \mathcal{F}$ with $\mathbb{P}(E) > 0$ and any G-cocycle A we have $\overline{\mathcal{E}}(A) = \overline{\mathcal{E}}(A_E)$ (see Schmidt [6, p. 21]).

We will show that the latter is in general false in the nonabelian case.

It follows immediately from the definition that

$$\overline{\mathcal{E}}(A) = \bigcap_{E \in \mathcal{F}; \mathbb{P}(E) > 0} \overline{\bigcup_{\omega \in E} \{\Phi_{A_E}(n, \omega) \mid n \in \mathbb{Z}\}},$$

hence

$$\overline{\mathcal{E}}(A) \subset \overline{\mathcal{E}}(A_E) \quad \text{for any } E \in \mathcal{F} \text{ with } \mathbb{P}(E) > 0.$$

We now prove that this inclusion can be strict.

THEOREM 3.1: (i) *There is a $Gl(2)$-cocycle A and a set $E \in \mathcal{F}$ with $\mathbb{P}(E) > 0$ such that $\overline{\mathcal{E}}(A)$ is a proper subgroup of $\overline{\mathcal{E}}(A_E)$.*
(ii) *There is an $SO(3)$-cocycle A and a set $E \in \mathcal{F}$ with $\mathbb{P}(E) > 0$ such that $\overline{\mathcal{E}}(A)$ is a proper subgroup of $\overline{\mathcal{E}}(A_E)$.*

Proof: We use the same idea as for the proof of Theorem 2.1.

(i) Choose any $E \in \mathcal{F}$ with $0 < \mathbb{P}(E) < 1$. Let A be as in the proof of Theorem 2.1. Choose a measurable map $D\colon \Omega \to SO(2)$ such that the function $\omega \mapsto D(\omega)e_1$, where e_1 is the point on the unit circle S^1 corresponding to the first standard basis vector, has support equal to S^1, and additionally $D(\omega) = I$ for all $\omega \in E$. This is possible because Ω is non-atomic. Put $B(\omega) = D(\theta\omega)A(\omega)D(\omega)^{-1}$ for all $\omega \in \Omega$. The same argument as in the proof of Theorem 2.1 implies that $\mathcal{E}(B) = \{I\}$. On the other hand, $B_E = A_E$ by the choice of D, which implies that $\mathcal{E}(B_E) = \mathcal{E}(A_E) \supset \mathcal{E}(A)$, hence $\mathcal{E}(B_E)$ is nontrivial. Therefore, $\mathcal{E}(B)$ is a proper subgroup of $\mathcal{E}(B_E)$, hence $\overline{\mathcal{E}}(B)$ is a proper subgroup of $\overline{\mathcal{E}}(B_E)$.

(ii) Use the construction of the proof of Theorem 2.2 and the same argument as in the proof of part (i) above. ∎

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